Suggested solution of HW3

Q2 Let $\epsilon > 0$ be such that $r = x + \epsilon < 1$. Then there exists $N \in \mathbb{N}$ such that for all $n > N$, $x_n^{1/n} < r$. In particular,

$$x_n < r^n \quad \forall\, n > N.$$

Result follows from letting $n \to \infty$.

Q3 (a)

$$\frac{n}{n^2} = \frac{1}{n} \to 0.$$

(b) Since

$$2^n = (1+1)^n \geq C_3^n = \frac{n(n-1)(n-2)}{6},$$

we have $n^2 2^{-n} \to 0$.

(c)

$$\frac{2^n}{100^n} = \left(\frac{1}{50}\right)^n \to 0.$$

(d) For n sufficiently large,

$$\frac{100^n}{n!} = \frac{100^{n-200}}{n \cdot (n-1)(n-2)...(200)} \cdot \frac{100^{200}}{200!} \leq \frac{100^{200}}{200!}\left(\frac{1}{2}\right)^{n-200} \to 0.$$

(e)

$$\frac{n!}{n^n} \leq \frac{1}{n} \to 0.$$

Q6 (Cauchy criterion) Let $\epsilon > 0$, we can find $N \in \mathbb{N}$, such that for all $m > n > N$,

$$0 < \sum_{k=n}^{m} y_k < \epsilon.$$

Hence, for the same ϵ, N, for all $m > n > N$,

$$\sum_{k=n}^{m} x_k < \epsilon.$$

Q7 Let $\epsilon > 0$, there exists N such that for all $m > n > N$,

$$\sum_{k=n}^{m} |x_k| < \epsilon.$$

Hence,

$$\left|\sum_{k=n}^{m} x_k\right| < \epsilon.$$

Result follows from cauchy criterion.